Lagrangian Multiform for the Rational Gaudin Model

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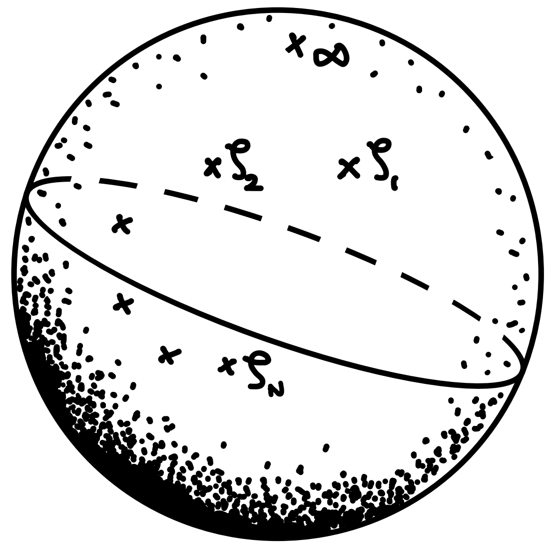

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Vincent Caudrelier, Marta Dell'Atti, AAS

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Outline

I Gaudin models } Introduction, history, motivation

II Lie dialgebras and Lax equations } Algebraic background

III Constructing Lagrangian multiforms on coadjoint orbits

IV Lagrangian multiform for the rational Gaudin model

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V Future directions } Connections, generalisations

I Gaudin models

Gaudin models

Gaudin models are a general class of integrable systems associated with **quadratic Lie algebras.**

Lie algebras with a nondegenerate invariant bilinear form

First introduced in the quantum finite-dimensional setup to describe **quantum spin chains.**
[Gaudin '76]

Various generalisations are known — corresponding to both finite- and infinite-dimensional algebras; and with rational, elliptic, skew-symmetric and non-skew-symmetric r-matrices.

A large class of non-ultralocal integrable field theories have been shown to be reinterpretations of **classical dihedral affine Gaudin models.**
[Vicedo '17]

Rational Gaudin models

The Lax matrix of a rational Gaudin model associated with a finite Lie algebra $\mathfrak{g}$ and a set of points $\zeta_r \in \mathbb{C}$ $(r = 1, \dots, N)$ and the point at infinity is given by

$$L(\lambda) = \sum_{r=1}^{N} \frac{X_r}{\lambda - \zeta_r} + X_\infty \,, \quad X_1, \dots, X_N, X_\infty \in \mathfrak{g} \,.$$

$\mathfrak{g}$-valued rational function in variable λ

The quadratic Gaudin Hamiltonians are given as

$$H_r = \sum_{s \neq r} \frac{\mathrm{Tr}(X_r X_s)}{\zeta_r - \zeta_s} + \mathrm{Tr}(X_r X_\infty), \quad r = 1, \dots, N \,.$$

describes long-range spin-spin interaction

But how would one describe Gaudin models
in the Lagrangian formalism?

II Lie dialgebras and Lax equations

Lie dialgebras

[Semenov-Tian-Shansky '83]

Let $\mathfrak{g}$ be a Lie algebra with a Lie bracket $[\,,\,]$, and $R: \mathfrak{g} \longrightarrow \mathfrak{g}$ be a linear map. If R is a solution of the **modified classical Yang-Baxter equation**

$$[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) = -[X, Y], \quad \forall X, Y \in \mathfrak{g},$$

then the skew-symmetric bracket

$$[X, Y]_R = \frac{1}{2}([R(X), Y] + [X, R(Y)])$$

satisfies the Jacobi identity and defines a second Lie algebra structure on $\mathfrak{g}$. We will denote the corresponding Lie algebra by $\mathfrak{g}_R$.

The pair $(\mathfrak{g}, \mathfrak{g}_R)$ is called a **Lie dialgebra**. not the same as a Lie bialgebra

Lie dialgebras

We now have another set of adjoint and coadjoint actions. For $\forall X, Y \in \mathfrak{g}$, $\forall \xi \in \mathfrak{g}^*$, we can define

$$\mathrm{ad}^R_X \cdot Y = [X,Y]_R \quad \text{and} \quad (\mathrm{ad}^{R*}_X \cdot \xi)Y = -\xi(\mathrm{ad}^R_X \cdot Y) = -\xi([X,Y]_R).$$

adjoint action of $\mathfrak{g}_R$ on $\mathfrak{g}$

coadjoint action of $\mathfrak{g}_R$ on $\mathfrak{g}^*$

We also have the following useful relation

$$R_+ - R_- = \mathrm{Id},$$

where $R_\pm = \frac{1}{2}(R \pm \mathrm{Id})$.

Let $\mathfrak{g}_\pm = \mathrm{Im}\, R_\pm$ and $X_\pm = R_\pm(X)$ for $X \in \mathfrak{g}$. One can show that for any element $X \in \mathfrak{g}$, we have a unique decomposition as

$$X = R_+(X) - R_-(X) = X_+ - X_-.$$

Lie dialgebras

Let us denote by G_R the Lie group associated with the Lie algebra $\mathfrak{g}_R$. The homomorphisms $R_\pm$ give rise to Lie group homomorphisms, which allow us to define the multiplication $\cdot_R$ in G_R as

$$g \cdot_R h = (g_+, g_-) \cdot_R (h_+, h_-) = (g_+ h_+, g_- h_-), \quad \forall g, h \in G_R,$$

where $g_\pm h_\pm$ denotes the product in G.

We have a new set of adjoint and coadjoint actions, those of G_R on $\mathfrak{g}_R$ and $\mathfrak{g}^*$, which we can denote in the following useful way:

$$\mathrm{Ad}^R_g \cdot X = \mathrm{Ad}_{g_+} \cdot X_+ - \mathrm{Ad}_{g_-} \cdot X_-, \quad \text{and}$$

$$\mathrm{Ad}^{R*}_g \cdot \xi = R^*_+ (\mathrm{Ad}_{g_+} \cdot \xi) - R^*_- (\mathrm{Ad}_{g_-} \cdot \xi), \quad \forall X \in \mathfrak{g}_R, \ \xi \in \mathfrak{g}^*, \ g \in G_R.$$

Lie-Poisson bracket and coadjoint orbits

Using the second Lie bracket on $\mathfrak{g}$, we can define an additional Lie-Poisson bracket on $\mathfrak{g}^*$, for $f, g \in C^\infty(\mathfrak{g}^*)$ and $\xi \in \mathfrak{g}^*$,

$$\{f, g\}_R(\xi) = (\xi, [\nabla f(\xi), \nabla g(\xi)]_R).$$

natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$: $\xi(X) = (\xi, X)$

the original Lie-Poisson bracket on $\mathfrak{g}^*$ reads $\{f, g\}(\xi) = (\xi, [\nabla f(\xi), \nabla g(\xi)])$

Its symplectic leaves are the coadjoint orbits of G_R in $\mathfrak{g}^*$.

We need one final ingredient: an Ad-invariant nondegenerate symmetric bilinear form $\langle\, , \rangle$ on $\mathfrak{g}$.

allows the identification of $\mathfrak{g}^*$ with $\mathfrak{g}$ and of the coadjoint actions with the adjoint actions

Involutivity theorem and Lax equations

The Ad^*-invariant functions on $\mathfrak{g}^*$ are in involution with respect to $\{\ ,\ \}_R$. The equation of motion

these function are simply Casimir functions with respect to $\{\ ,\ \}$

$$\frac{d}{dt} L = \{L, H\}_R$$

induced by an Ad^*-invariant function H on $\mathfrak{g}^*$ takes the following equivalent forms, for an arbitrary $L \in \mathfrak{g}^*$,

$$\frac{d}{dt} L = ad^{R*}_{\nabla H(L)} \cdot L = \frac{1}{2}\, ad^*_{R\nabla H(L)} \cdot L = ad^*_{R_\pm \nabla H(L)} \cdot L .$$

using $\{\ ,\ \}$ would have given trivial equations

Involutivity theorem and Lax equations

The Ad-invariant nondegenerate symmetric bilinear form $\langle\, , \rangle$ on $\mathfrak{g}$ allows us to rewrite the last equation in the form of a Lax equation

$$\frac{d}{dt} L = [M_\pm, L], \quad M_\pm = R_\pm \nabla H(L).$$

So, the natural arena to define our phase space is a coadjoint orbit of G_R in $\mathfrak{g}^*$,

$$\mathcal{O}_\Lambda = \{ \mathrm{Ad}^{R*}_\varphi \cdot \Lambda ;\ \varphi \in G_R \}, \quad \Lambda \in \mathfrak{g}^*.$$

this is where the Lax matrix L lives

!

Takeaway message

Special case: the Adler-Kostant-Symes scheme
[Adler '78], [Symes '78], [Kostant '79]

One gets the well-known **Adler-Kostant-Symes scheme** by fixing Λ to be in $\mathfrak{g}_-^*$.

This choice results in only the subgroup G_- in $G_R \simeq G_+ \times G_-$ playing a role since

$$L = \mathrm{Ad}_{\varphi}^{R*} \cdot \Lambda = -R_-^*(\mathrm{Ad}_{\varphi_-}^* \cdot \Lambda).$$

Thus, the coadjoint orbit $\mathcal{O}_\Lambda$ lies in $\mathfrak{g}_-^*$.

On to the multi-time story now!

Compatible time flows

For any two Ad^*-invariant functions H_1 and H_2 on $\mathfrak{g}^*$, we have

$$\{H_1, H_2\}_R = 0.$$

This means that if we have a sufficient number of such independent functions, we can define compatible time flows associated with a family of Ad^*-invariant functions H_k, $k = 1, \dots, N$.

We then obtain an integrable hierarchy with equations in Lax form

$$\partial_{t_k} L = [R_\pm \nabla H_k(L), L], \quad k = 1, \dots, N.$$

But how would one capture these integrable hierarchies in the Lagrangian formalism?

III Constructing Lagrangian multiforms on coadjoint orbits

The general Lagrangian multiform

[Caudrelier - Dell'Atti - Singh '23]

We introduce the Lagrangian 1-form

$$\mathcal{L}[\varphi] = \sum_{k=1}^{N} \mathcal{L}_k \, dt_k = \mathcal{K}[\varphi] - \mathcal{H}[\varphi]$$

with kinetic part

$$\mathcal{K}[\varphi] = \sum_{k=1}^{N} \left(L, \partial_{t_k} \varphi \cdot_R \varphi^{-1} \right) dt_k, \qquad L = \mathrm{Ad}^{R*}_{\varphi} \cdot \Lambda, \qquad \varphi \in G_R,$$

fixed non-dyamical element of $\mathfrak{g}^*$ defining the phase space $\mathcal{O}_\Lambda$

field containing the dynamical degrees of freedom of the system

and potential part

$$\mathcal{H}[\varphi] = \sum_{k=1}^{N} H_k(L) \, dt_k .$$

Ad^*-invariant functions $H_k \in C^\infty(\mathfrak{g}^*)$

Euler-Lagrange equations = Lax equations // Result I

[Caudrelier - Dell'Atti - Singh '23]

On considering the variation of the Lagrangian 1-form $\mathcal{L}$, we can derive the Euler-Lagrange equations which take the form

$$\partial_{t_k} L = \frac{1}{2} \, ad^*_{R \nabla H_k(L)} \cdot L , \qquad k = 1, \dots, N .$$

Then, on identifying $\mathfrak{g}^*$ with $\mathfrak{g}$, and ad^* with ad, we get

$$\partial_{t_k} L = [R_\pm \nabla H_k(L), L], \qquad k = 1, \dots, N,$$

which is exactly the Lax equation associated with the Lax matrix L.

Closure relation // Result II

[Caudrelier - Dell'Atti - Singh '23]

Next, we establish the closure relation for the Lagrangian 1-form $\mathcal{L}$, that is,

$$d\mathcal{L} = 0 \,, \qquad \text{on shell},$$

or equivalently,

$$\partial_{t_j} \mathcal{L}_k - \partial_{t_k} \mathcal{L}_j = 0 \,, \qquad \text{on shell}.$$

This is a consequence of the Ad^*-invariance of H and the fact that R is a solution of the modified CYBE.

Closure relation and Poisson involutivity // Result III

[Caudrelier - Dell'Atti - Singh '23]

Further, for the Lagrangian 1-forms in this class, we can prove

$$\frac{\partial \mathcal{L}_k}{\partial t_\ell} - \frac{\partial \mathcal{L}_\ell}{\partial t_k} = \{H_k, H_\ell\}_R = 0, \qquad \text{on shell},$$

demonstrating the connection between the closure relation for Lagrangian 1-forms and the involutivity of Hamiltonians.

first established in [Suris '13]

This is, in fact, a corollary of a deeper structural result proved in [Caudrelier - Dell'Atti - Singh '23].

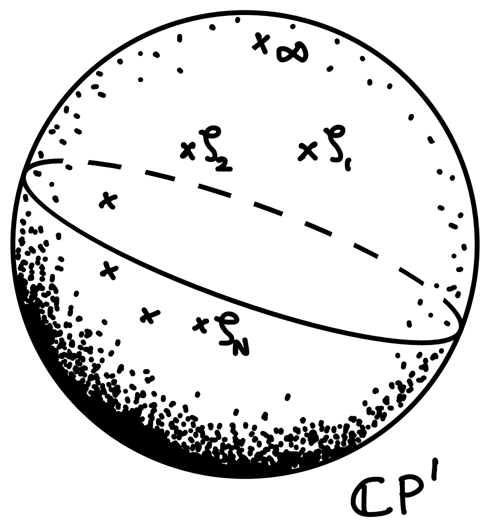
IV Lagrangian multiform for the rational Gaudin model

Rational Gaudin model

The Lax matrix of a rational Gaudin model associated with a finite Lie algebra $\mathfrak{g}$ and a set of points $\xi_r \in \mathbb{C}$ $(r = 1, \dots, N)$ and the point at infinity is given by

$$L(\lambda) = \sum_{r=1}^{N} \frac{X_r}{\lambda - \xi_r} + X_\infty \,, \quad X_1, \dots, X_N, X_\infty \in \mathfrak{g} \,,$$

$\mathfrak{g}$-valued rational function in variable λ



with the corresponding Lax equations

$$\partial_{t_i^r} X_s = \frac{[X_r, X_s]}{\xi_r - \xi_s} \,, \quad s \neq r \,,$$

$$\partial_{t_i^r} X_r = -\sum_{s \neq r} \frac{[X_r, X_s]}{\xi_r - \xi_s} - [X_r, X_\infty] \,, \qquad \partial_{t_i^r} X_\infty = 0 .$$

Algebraic setup

We need to choose a suitable Lie algebra and a linear map from the Lie algebra to itself.

Let us fix

$Q = \{\zeta_1, \ldots, \zeta_N, \infty\} \subset \mathbb{C}P^1,$ these become the sites of the model

a finite set of points in $\mathbb{C}P^1$ including the point at infinity,

and an index set $S = \{1, \ldots, N, \infty\}$.

Denote by

$\mathfrak{g}_Q$ the algebra of $\mathfrak{g}$-valued rational function

in the formal variable λ with poles in Q. this is where the Lax matrix lives

Define the local parameters $\lambda_r = \lambda - \zeta_r, \quad \zeta_r \neq \infty, \quad$ and $\lambda_\infty = \dfrac{1}{\lambda}$.

Algebraic setup

Define the direct sum of Lie algebras

$$\tilde{\mathfrak{g}}_Q = \bigoplus_{r \in S} \mathfrak{g}_r$$

the Lie algebra we will work with

where

$$\tilde{\mathfrak{g}}_r = \mathfrak{g} \otimes \mathbb{C}((\lambda_r))$$

is the algebra of formal Laurent series in variable λ_r with coefficients in $\mathfrak{g}$, and Lie bracket

$$[X\lambda_r^i, Y\lambda_r^j] = [X, Y]\lambda_r^{i+j}, \quad X, Y \in \mathfrak{g}.$$

elements of $\tilde{\mathfrak{g}}_Q$ are tuples $(X_1(\lambda_1), \ldots, X_N(\lambda_N), X_\infty(\lambda_\infty))$ with $X_1, \ldots, X_N, X_\infty \in \mathfrak{g}$

Algebraic setup

We can define a vector space decomposition of $\tilde{\mathfrak{g}}_Q$ into Lie subalgebras

$$\underbrace{\tilde{\mathfrak{g}}_Q = \tilde{\mathfrak{g}}_{Q+} \oplus \tilde{\mathfrak{g}}_{Q-}}$$

we will denote by $P_\pm$ the projectors determined by this decomposition

with $\tilde{\mathfrak{g}}_{Q\pm} = \bigoplus_{\mathrm{res}} \mathfrak{g}_{r\pm}$

where

$$\tilde{\mathfrak{g}}_{r+} = \mathfrak{g} \otimes \mathbb{C}[[\lambda_r]], \quad r \neq \infty,$$

algebra of formal Taylor series in λ_r

$$\tilde{\mathfrak{g}}_{\infty+} = \mathfrak{g} \otimes \lambda_\infty \mathbb{C}[[\lambda_\infty]],$$

algebra of formal Taylor series in λ_∞ without the constant term

and

$$\tilde{\mathfrak{g}}_{r-} = \mathfrak{g} \otimes \lambda_r^{-1} \mathbb{C}[\lambda_r^{-1}], \quad r \neq \infty,$$

algebra of polynomials in λ_r^{-1} without the constant term

$$\tilde{\mathfrak{g}}_{\infty-} = \mathfrak{g} \otimes \mathbb{C}[\lambda_\infty^{-1}].$$

algebra of polynomials in λ_∞^{-1}

Algebraic setup

Further, we have an embedding of Lie algebras

$$\iota_\lambda : \mathcal{Y}_Q(\mathfrak{g}) \hookrightarrow \tilde{\mathfrak{g}}_Q , \qquad f \longmapsto (\iota_{\lambda_1} f, \dots, \iota_{\lambda_N} f, \iota_{\lambda_\infty} f),$$

maps $f \in \mathcal{Y}_0(\mathfrak{g})$ to the tuple of its Laurent expansion at points $\zeta_1, \dots, \zeta_N, \zeta_\infty$

which induces the vector space decomposition

$$\tilde{\mathfrak{g}}_Q = \tilde{\mathfrak{g}}_{Q+} \oplus \iota_\lambda \mathcal{Y}_Q(\mathfrak{g}).$$

We will denote by $\Pi_\pm$ the projectors corresponding to this decomposition.

not the same as $P_\pm$

The r-matrix we need is

$$R = \Pi_+ - \Pi_- .$$

we will use it to equip $\tilde{\mathfrak{g}}_Q$ with a dialgebra structure

Algebraic setup

To identify the dual space to $\mathcal{C}_\Lambda \mathcal{Y}_Q(\mathfrak{g})$, we will use the nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$,

$$(X, Y) \longmapsto \mathrm{Tr}(XY),$$

to define a nondegenerate invariant symmetric bilinear form on $\tilde{\mathfrak{g}}_Q$:

$$\langle X, Y \rangle = \sum_{r \in S} \operatorname*{Res}_{\lambda_r = 0} \mathrm{Tr}\left(X_r(\lambda_r) Y_r(\lambda_r)\right),$$

which induces the decomposition

$$\tilde{\mathfrak{g}}_Q^* = \tilde{\mathfrak{g}}_{Q-}^* \oplus \tilde{\mathfrak{g}}_{Q+}^* \simeq \tilde{\mathfrak{g}}_{Q+}^\perp \oplus \tilde{\mathfrak{g}}_{Q-}^\perp .$$

Algebraic setup

Both $\tilde{\mathfrak{g}}_{Q+}$ and $\iota_\lambda \mathcal{Y}_Q(\mathfrak{g})$ are (maximally) isotropic to this bilinear form, which allows us to make the identification

elements of this are those we need to work with

$$\tilde{\mathfrak{g}}^*_{Q+} \simeq \iota_\lambda \mathcal{Y}_Q(\mathfrak{g}).$$

So, coadjoint orbits of $\tilde{G}_{Q+}$ in $\tilde{\mathfrak{g}}^*_{Q+}$ will be the phase space where the Lax matrix of the model lives and where we will describe its dynamics.

elements of $\tilde{G}_{Q+}$ are of the form

$$\varphi_+ = (\varphi_{1+}(\lambda_1), \dots, \varphi_{N+}(\lambda_N), \varphi_{\infty+}(\lambda_\infty))$$

with $\varphi_{r+}(\lambda_r) = \sum_{n=0}^{\infty} \phi_r^{(n)} \lambda_r^n$

and $\varphi_{\infty+}(\lambda_r) = 1 + \sum_{n=1}^{\infty} \phi_\infty^{(n)} \lambda_\infty^n$

Algebraic setup

The coadjoint orbit of an element $c_\lambda f \in \tilde{g}^*_{Q+}$ is given by

$$
\begin{aligned}
c_\lambda F &= Ad^{R*}_{\varphi} \cdot c_\lambda f \\
&= R^*_+ (Ad^*_{\varphi_+} \cdot c_\lambda f) \\
&= R^*_+ (\varphi_+ \cdot c_\lambda f \cdot \varphi_+^{-1}) \\
&= \Pi_- (\varphi_+ \cdot c_\lambda f \cdot \varphi_+^{-1})
\end{aligned}
$$

since we are looking at an element from a subspace of the dual only one corresponding subgroup plays a role in the coadjoint orbit

where we have made the identification $R^*_+ = \Pi_-$.

So, we are now ready with our setup!

Lax matrix

Choose

$$\Lambda(\lambda) = \sum_{r=1}^{N} \frac{\Lambda_r}{\lambda - \zeta_r} + \Omega , \qquad \Lambda_r, \Omega \in \mathfrak{g}$$

and consider its embedding into $\tilde{\mathfrak{g}}_Q$

$$\iota_\lambda \Lambda(\lambda) = \iota_\lambda \left(\sum_{r=1}^{N} \frac{\Lambda_r}{\lambda - \zeta_r} + \Omega \right) \in \iota_\lambda \mathcal{Y}_Q(\mathfrak{g}) \simeq \tilde{\mathfrak{g}}^*_{Q+}.$$

The orbit of $\iota_\lambda \Lambda$ under the coadjoint action of $\tilde{G}_{Q+}$ will be

$$\iota_\lambda L = \Pi_- \left(\varphi_+ \cdot \iota_\lambda \Lambda \cdot \varphi_+^{-1} \right)$$

contains the dynamical degrees of freedom

$$= \iota_\lambda \left(\sum_{r=1}^{N} \frac{A_r}{\lambda - \zeta_r} + \Omega \right).$$

fixed non-dynamical element

$A_r = \Phi_r^{(0)} \Lambda_r \Phi_r^{(0)-1}$

Lagrangian multiform for the rational Gaudin model

[Caudrelier - Dell'Atti - Singh '23]

We can now write down the Gaudin multiform on the orbit of $\Lambda(\lambda)$, with the elements $\iota_\lambda L$,

$$\mathcal{L} = \sum_{k=1}^{N} \sum_{r\in S} \mathcal{L}_{k,r}\, dt_k^r ,$$

with

restriction of $H_{k,r}$ to $\iota_\lambda L$

$$\mathcal{L}_{k,r} = \sum_{s\in S} \operatorname*{Res}_{\lambda_s=0} \operatorname{Tr}\left(\iota_{\lambda_s} L\, \partial_{t_k^r} \varphi_{s+}(\lambda_s)\varphi_{s+}(\lambda_s)^{-1}\right) - H_{k,r}(\iota_\lambda L).$$

Upon simplification, the Lagrangian coefficients take the form

$$\mathcal{L}_{k,r} = \sum_{s=1}^{N} \operatorname{Tr}\left(\Lambda_s\, \phi_s^{-1}\partial_{t_k^r}\phi_s\right) - H_{k,r}(\iota_\lambda L).$$

$\phi_s^{(0)} = \phi_s$ for notational simplicity

Lagrangian multiform for the rational Gaudin model

[Caudrelier - Dell'Atti - Singh '23]

The potential part $H_{k,r}(\iota_\lambda L)$ is the restriction to $\iota_\lambda L$ of invariant functions on $\tilde{\mathfrak{g}}_Q$ that can be given by

$$H_{k,r} : X \in \tilde{\mathfrak{g}}_Q \longmapsto \operatorname*{Res}_{\lambda_r=0} \frac{\mathrm{Tr}\left(X_r(\lambda_r)^{k+1}\right)}{k+1}, \quad k \geqslant 1 .$$

For $k=1, 2$, we have

$$H_{1,r}(\iota_\lambda L) = \sum_{s\neq r} \frac{\mathrm{Tr}(A_r A_s)}{\zeta_r - \zeta_s} + \mathrm{Tr}(A_r \Omega)$$

and

$$H_{2,r}(\iota_\lambda L) = \mathrm{Tr}\left(A_r \left(\sum_{s\neq r} \frac{A_s}{\zeta_r - \zeta_s} + \Omega \right)^2 \right) - \mathrm{Tr}\left(A_r^2 \left(\sum_{s\neq r} \frac{A_s}{(\zeta_r - \zeta_s)^2} \right)\right).$$

Euler-Lagrange equations

Varying $\mathcal{L}_{1,r}$ and $\mathcal{L}_{2,r}$ with respect to ϕ_s, $s=1,\dots,N$, gives the Euler-Lagrange equations for the first and the second time flows respectively:

$$\partial_{t_1^r} A_s = \frac{[A_r, A_s]}{\zeta_r - \zeta_s}, \quad s \neq r,$$

$$\partial_{t_1^r} A_r = -\sum_{s\neq r} \frac{[A_r, A_s]}{\zeta_r - \zeta_s} - [A_r, \Omega],$$

$$\partial_{t_2^r} A_s = \frac{-[A_r^2, A_s]}{(\zeta_r - \zeta_s)^2} + \sum_{s'\neq r} \frac{[A_r A_{s'} + A_{s'} A_r, A_s]}{(\zeta_r - \zeta_s)(\zeta_r - \zeta_{s'})} + \frac{[A_r \Omega + \Omega A_r, A_s]}{\zeta_r - \zeta_s}, \quad s \neq r,$$

$$\partial_{t_2^r} A_r = \sum_{s\neq r} \frac{[A_r^2, A_s]}{(\zeta_r - \zeta_s)^2} - \sum_{s\neq r}\sum_{s'\neq r} \frac{[A_r, A_s A_{s'}]}{(\zeta_r - \zeta_s)(\zeta_r - \zeta_{s'})} - \sum_{s\neq r} \frac{[A_r, A_s \Omega + \Omega A_s]}{\zeta_r - \zeta_s} - [A_r, \Omega^2].$$

V Future directions

Future directions

I Lagrangian multiform for cyclotomic Gaudin models
[work in progress with V. Caudrelier and B. Vicedo]

non-skew-symmetric r-matrix — using the full power of Lie dialgebras

II Lagrangian multiform for affine Gaudin models

reinterpretation as non-ultralocal integrable field theories

?

quantisation

III Connections of the Lie dialgebra construction with the gauge-theoretic approach to integrability, in particular, mixed BF theory-based construction of Gaudin models
[cf. B. Vicedo's talk]

A not-at-all exhaustive list of references

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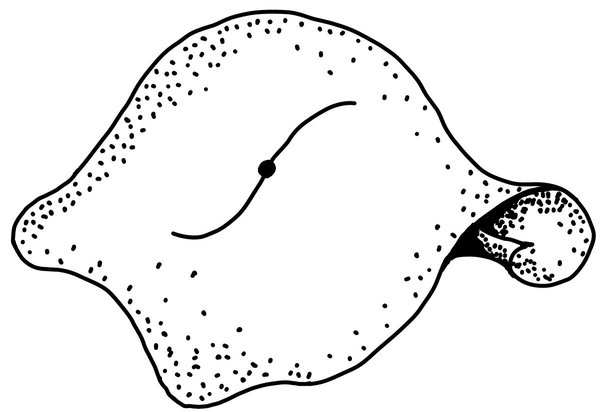
Thank you!



Hamiltonians : the traditional approach to integrability

A $2N$-dimensional Hamiltonian system is (Liouville) integrable if it possesses N independent conserved quantities H_j in Poisson involution, that is,

$$\{H_i, H_j\} = 0 .$$

One of the H_i can be taken as the Hamiltonian of interest H.

This gives us the notion of an **integrable hierarchy**: each H_k can be used define a dynamical system each with respect to a "time" variable t_k.

Hamiltonians : the traditional approach to integrability

We have a hierarchy of commuting Hamiltonian flows:

$$\underbrace{\{H_j, H_k\} = 0}_{\text{Poisson involutivity of Hamiltonians}} \Rightarrow \underbrace{\left[\frac{\partial}{\partial t_j}, \frac{\partial}{\partial t_k}\right] = 0}_{\text{commutativity of vector fields}} .$$

This implies path-independence in the **multi-time** $(t_1, \dots, t_N)$ space.

Think not of a single integrable system,
but of the entire hierarchy it lives in.

! Takeaway message

Lagrangian multiforms: a variational criterion for integrability

A variational criterion for integrability was introduced in [Lobb-Nijhoff '09] in a discrete setup.

What we need is a collection of Lagrangians $\mathcal{L}_k$ associated with times t_k assembled into a 1-form

$$\mathcal{L}[q] = \sum_{k=1}^{N} \mathcal{L}_k[q]\, dt_k .$$

central objects in the Lagrangian multiform theory for finite-dimensional integrable systems

Here q denotes generic configuration coordinates. By $\mathcal{L}[q]$ and $\mathcal{L}_k[q]$, we mean that these quantities depend on q and a finite number of derivatives of q with respects to the times $t_1, \dots, t_N$.

Lagrangian multiforms: a variational criterion for integrability

We now have an associated generalised action

$$S[q,\Gamma] = \int_\Gamma \mathcal{L}[q]$$

this replaces the traditional action $S[q] = \int_a^b L[q]\,dt$

where Γ is a curve in the multi-time $\mathbb{R}^N$ with coordinates $t_1, \dots, t_N$.
Applying the generalised variational principle to $\mathcal{L}$ gives the multi-time Euler-Lagrange equations

$$\frac{\partial \mathcal{L}_k}{\partial q} - \partial_{t_k} \frac{\partial \mathcal{L}_k}{\partial q_{t_k}} = 0,$$

standard Euler-Lagrange equation for each $\mathcal{L}_k$

New (corner) Euler-Lagrange equations

$$\left\{\begin{aligned} &\frac{\partial \mathcal{L}_k}{\partial q_{t_\ell}} = 0, \qquad \ell \neq k, \\ &\frac{\partial \mathcal{L}_k}{\partial q_{t_k}} = \frac{\partial \mathcal{L}_\ell}{\partial q_{t_\ell}}, \quad k,\ell = 1,\dots,N. \end{aligned}\right.$$

Lagrangian coefficient $\mathcal{L}_k$ cannot depend on velocities q_{t_ℓ} for $\ell \neq k$

conjugate momentum to q is the same with respect to all times t_k

Lagrangian multiforms: a variational criterion for integrability

On the solutions of the multi-time Euler-Lagrange equations, we require

$$S[q, \Gamma] = S[q, \Gamma']$$

for all curves Γ, Γ' in the multi-time space.

This implies the closure relation

$$d\mathcal{L}[q] = 0 \iff \partial_{t_k} \mathcal{L}_j - \partial_{t_j} \mathcal{L}_k = 0$$

equivalent to the Poisson involutivity of Hamiltonians

on shell.